Motivation: Turn a hard problem (equation containing derivatives & integrals) into an equivalent but easy problem (algebraic equation.)

Definition:
\[
L(f(t)) = \hat{f}(s) = \int_0^\infty e^{-st} f(t) \, dt
\]

Transforming Constants:
\[
L(1) = \int_0^\infty e^{-st} \cdot 1 \, dt = \int_0^\infty e^{-st} \, dt = \left[ -\frac{e^{-st}}{s} \right]_0^\infty = \frac{1}{s}
\]

Transforming Exponentials:
\[
L(e^{\alpha t}) = \int_0^\infty e^{-st} e^{\alpha t} \, dt = \int_0^\infty e^{-(s-\alpha)t} \, dt = \left[ -\frac{e^{-(s-\alpha)t}}{s-\alpha} \right]_0^\infty = \frac{1}{s-\alpha}
\]
Transforming

\[ L \{ f(t) \} = \int_{0}^{\infty} e^{-st} f(t) \, dt \]

Derivatives:

Integrate by Parts:

\[ \int u \, dv = uv - \int v \, du \]

\[ u = e^{-st} \quad dv = f(t) \, dt \]
\[ du = -se^{-st} \quad v = f(t) \]

\[ = e^{-st} f(t) \bigg|_{0}^{\infty} + s \int_{0}^{\infty} e^{-st} f(t) \, dt \]

\[ = -f(0) + s \int f(t) \, dt \]

\[ = s \hat{f}(s) - f(0) \]

All we have to do is multiply by \( s \)!

Transforming

Integrals:

\[ L \{ \int_{0}^{\infty} f(t) \, dt \} = \frac{\hat{f}(s)}{s} \]

Integrals are just anti-derivatives.
"Undoing" Laplace Transforms

In theory: \[ f(t) = \int_{-\infty}^{\infty} \frac{1}{2\pi i} \hat{f}(s) e^{st} \, ds \]

But as this involves complex integration, we will not apply it.

In practice: Build a table of \( f(t) \) and their corresponding \( \hat{f}(s) \), and lookup.

Example: Partial fractions

Find the \( f(t) \) that, when Laplace transformed, gives

\[ \hat{f}(s) = \frac{2s - 14}{s^2 - 2s - 3} \]

Problem takes the form \( \frac{\text{polynomial}}{\text{polynomial}_2} \) \( \Rightarrow \) apply method of partial fractions

1. Factor denominator:

\[ \hat{f}(s) = \frac{2s - 14}{(s+1)(s-3)} \]

2. Express as sum of two fractions with unknown numerators:

\[ \hat{f}(s) = \frac{A}{s+1} + \frac{B}{s-3} \]

3. Multiply by common denominator:

\[ \hat{f}(s) = \frac{A(s-3) + B(s+1)}{(s+1)(s-3)} \]

4. Solve for \( A, B \)

\[ A(s-3) + B(s+1) = 2s - 14 \]
\[ As + Bs = 2s \]
\[ -3A + B = -14 \]
\[ \Rightarrow A = 4, \ B = -2 \]

\[ \hat{f}(s) = \frac{4}{s+1} - \frac{2}{s-3} \]
Look up each partial fraction:

\[ f(t) = 4 \mathcal{L}^{-1} \left[ \frac{1}{s+1} \right] - 2 \mathcal{L}^{-1} \left[ \frac{2}{s-3} \right] \]

\[ = 4e^{-t} - 2e^{3t} \]

**Analyzing Long-term behavior \((t \to \infty)\) of \(f(t)\) & \(\tilde{f}(s)\)**

For \(f(t) = 4e^{-t} - 2e^{3t}\), we see that the \(e^{3t}\) term will dominate, so \(\lim_{t \to \infty} f(t) = -\infty\).

We can predict the same behavior by looking at \(\tilde{f}(s)\):

\[ \tilde{f}(s) = \frac{2s-14}{(s+1)(s-3)} \]

At \(s = -1\) & denominator = 0

\(s = 3\), "blow-up"

\(\Rightarrow\) poles of \(\tilde{f}(s)\)

In general, \(s = \delta + iw\) can be complex. Plot \(s\) in complex plane:

- \(\text{In}(s)\)
- \((0,\infty)\)
- \(\text{Re}(s)\)

When \(\delta > 0\), \(f(t)\) goes to \(\infty\) or \(-\infty\) in the long run

Stable \(\uparrow\) Unstable

Pure Oscillations

**Final Value Theorem:** When all poles are on the left side of the complex plane:

\[ \lim_{t \to \infty} f(t) = \lim_{s \to 0} s \tilde{f}(s) \]