A Brief Linear Algebra Tutorial

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Suppose that our system of interest can be written, after the appropriate de-dimensionalization, as:

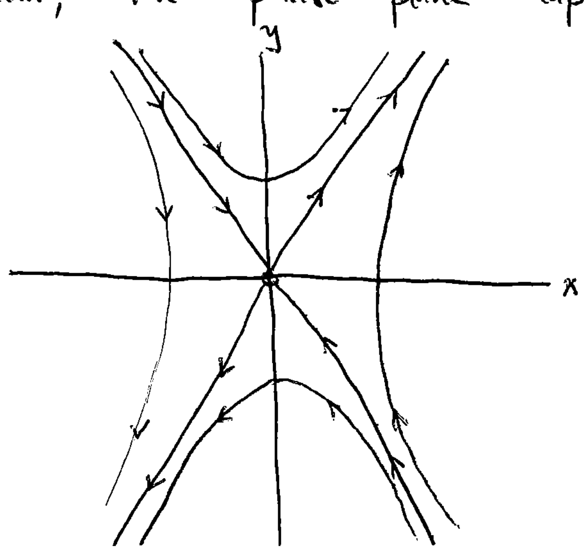
$$\frac{dx}{dt} = x + y$$

$$\frac{dy}{dt} = 4x + y$$

Where x and y are our dynamical variables of interest. We want to know certain properties of this system such as:

1) For a given initial condition (x_0, y_0) at $t=0$, what will happen to x and y as t increases?
2) Are there any places where x and y are at equilibrium (doesn't change over time)?
3) How sensitive these equilibrium points are to slight perturbations (stability analysis).

One way to summarize this information is by drawing the <u>phase plane</u> of the system. The phase plane describes how given points (x, y) change over time. For the above system, the phase plane happens to look like:

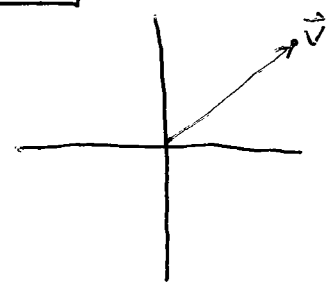


The goal of this tutorial is to understand how we go from the equations to the picture.

First some important concepts :

Vectors and How to Represent Them

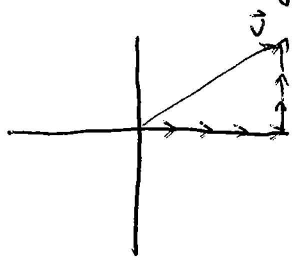
Suppose we have a vector $\vec{v}$ in a plane :



In order to specify where $\vec{v}$ is, we give it a pair of coordinates :

$$\vec{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

The numbers 4 and 3 tell you where $\vec{v}$ is relative to some set of coordinate system, in this case the x and y axes. In particular, it means that $\vec{v}$ is composed of 4 vectors of length 1 in the x-axis, and 3 in the y-axis :

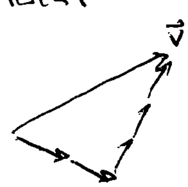


where $\hat{x} = \rightarrow$, $\hat{y} = \uparrow$ or $\vec{v} = 4\hat{x} + 3\hat{y}$

Here the vectors $\hat{x}$ and $\hat{y}$ constitute a basis in which $\vec{v}$ is represented. But $\hat{x}$ and $\hat{y}$ are an arbitrary choice. We could choose a different set of basis vectors $\vec{a}$ and $\vec{b}$:

$\vec{a} = \searrow$

$\vec{b} = \nearrow$



or $\vec{v} = 2\vec{a} + 3\vec{b}$

In the coordinate system where $\vec{a}$ and $\vec{b}$ are the basis vectors, $\vec{v}$ could be represented as the pair of numbers $(2, 3)$.

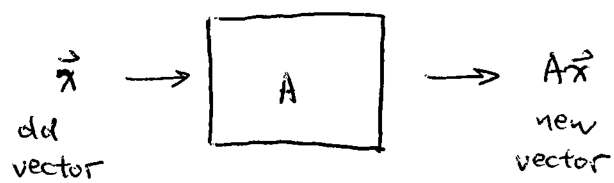
Bottom line : the numbers used to represent vectors (and matrices) depend on the coordinate system.

Matrices and Linear Transformations

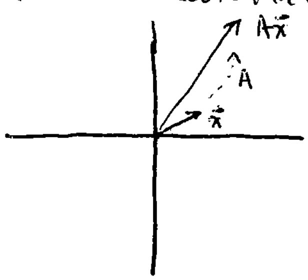
We have some intuitive sense that vectors tell us about location: where we are and where we're going. What about matrices? The intuition is that a matrix times a vector gives another vector:

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}}_{\substack{A \\ (2\times 2 \text{ matrix})}} \underbrace{\begin{bmatrix} 4 \\ 3 \end{bmatrix}}_{\substack{\vec{x} \\ (2\times 1 \text{ vector})}} = 4\begin{bmatrix} 1 \\ 4 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 7 \\ 19 \end{bmatrix}}_{\substack{A\vec{x} \\ (2\times 1 \text{ vector})}}$$

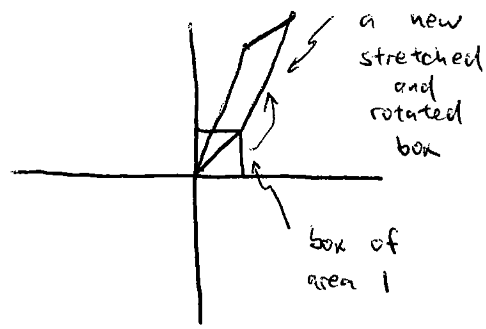
So a matrix represents a <u>rule</u> for changing a vector to another vector. Such changes are <u>linear transformations</u>.



In the coordinate plane:



Geometrically, the role of A is to <u>stretch</u> $\vec{x}$ and <u>rotate</u> it a bit counter-clockwise



There are two special linear transforms:

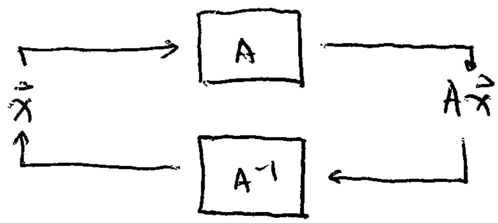
1) The inverse of A, denoted A^{-1}

e.g. $\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}^{-1} = \frac{1}{3}\begin{bmatrix} -1 & 1 \\ 4 & -1 \end{bmatrix}$ (Don't worry how to calculate inverses for now)

$$\underbrace{\frac{1}{3}\begin{bmatrix} -1 & 1 \\ 4 & -1 \end{bmatrix}}_{A^{-1}} \underbrace{\begin{bmatrix} 7 \\ 19 \end{bmatrix}}_{A\vec{x}} = \frac{1}{3}\begin{bmatrix} -7+19 \\ 28-19 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$A^{-1}(A\vec{x}) = \vec{x}$$

A^{-1} is the linear transform that exactly "undoes" the effect of A, such that $A^{-1}A\vec{x} = \vec{x}$. In our "black box" picture:



Caution: Not all matrices are invertible; sometimes a linear transformation is a one-way street - you can't go back the way you came!

2) The identity matrix is the linear transform that does nothing to $\vec{x}$: i.e. $I\vec{x} = \vec{x}$. I has 1's along the diagonal and zeroes elsewhere

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I} \underbrace{\begin{bmatrix} 4 \\ 3 \end{bmatrix}}_{\vec{x}} = 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 4 \\ 3 \end{bmatrix}}_{\vec{x}}$$

In particular, note that $A^{-1}A = AA^{-1} = I$

$$\underbrace{\frac{1}{3}\begin{bmatrix} -1 & 1 \\ 4 & -1 \end{bmatrix}}_{A^{-1}} \underbrace{\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}}_{A} = \frac{1}{3}\begin{bmatrix} (-1)(1)+(1)(4) & (-1)(1)+(1)(1) \\ (4)(1)+(-1)(4) & (4)(1)+(-1)(1) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I}$$

Equipped with this knowledge of linear transformations, let's go back to this notion of vectors in different coordinate systems. In particular, we want to be able to do the following: given a vector's coordinates in one set of axes, how do we find its coordinates in a different set of axes?

We have a vector $\vec{v}$ in a plane whose coordinates in the standard x, y coordinate system is $(6, 4)$. That is to say:

$$\vec{v} = 6 \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\hat{x}} + 4 \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\hat{y}}$$

where $\hat{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\hat{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are the basis vectors for the standard coordinate system.

We want to now find the coordinates of $\vec{v}$ in a new coordinate system defined by the new basis vectors

$$\vec{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \vec{b}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

That is, how many $\vec{b}_1$'s and $\vec{b}_2$'s does it take to get to $\vec{v}$? Or:

$$\beta_1 \vec{b}_1 + \beta_2 \vec{b}_2 = \vec{v}$$

we could solve for β_1, β_2

$$\beta_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}}_{\vec{v}_{new}} = \underbrace{\begin{bmatrix} 6 \\ 4 \end{bmatrix}}_{\vec{v}}$$

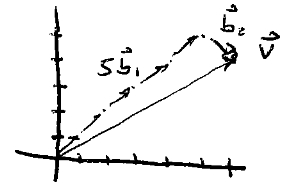
Look carefully at what this equation is saying: it tells you that if you already have the new coordinates of $\vec{v}$, you can multiply it by a matrix P to transform <u>back</u> to the old coordinates. P is a matrix that specifies the rules for changing coordinate systems and is comprised of the vectors of the new basis

$$P = \begin{bmatrix} \uparrow & \uparrow \\ \vec{b}_1 & \vec{b}_2 \\ \downarrow & \downarrow \end{bmatrix}$$

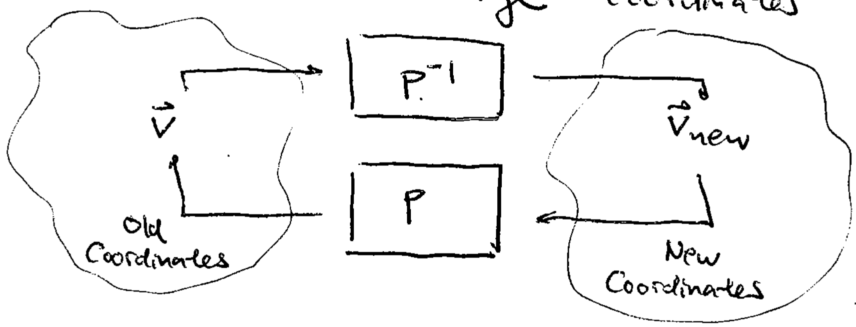
This suggests that to go from the old to new coordinates, we should multiply $\vec{v}$ by P inverse:

$$\underbrace{\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{P^{-1}} \underbrace{\begin{bmatrix} 6 \\ 4 \end{bmatrix}}_{\vec{v}} = \frac{1}{2}\begin{bmatrix} 6+4 \\ 6-4 \end{bmatrix} = \underbrace{\begin{bmatrix} 5 \\ 1 \end{bmatrix}}_{\vec{v}_{new}}$$

So $\vec{v} = 5\vec{b}_1 + 1\vec{b}_2$. Geometrically:



To summarize how to change coordinates



Now that we know how to change coordinate systems, this begs the questions: 1) Why would we ever want to change coordinates? and 2) How do we know what coordinates to change to?

As a motivating example, let's go back to our original system of D.E.'s:

$$\boxed{\begin{aligned} \frac{dx}{dt} &= x + y \\ \frac{dy}{dt} &= 4x + y \end{aligned}}$$

This system is difficult to solve because $\frac{dx}{dt}$ depends on y, and $\frac{dy}{dt}$ depends on x. These cross-terms mean we can't just separate variables and integrate. This system is said to be "<u>coupled</u>".

The way to approach this problem is to decouple it. As it turns out, if we work in a different coordinate system, the DE's become:

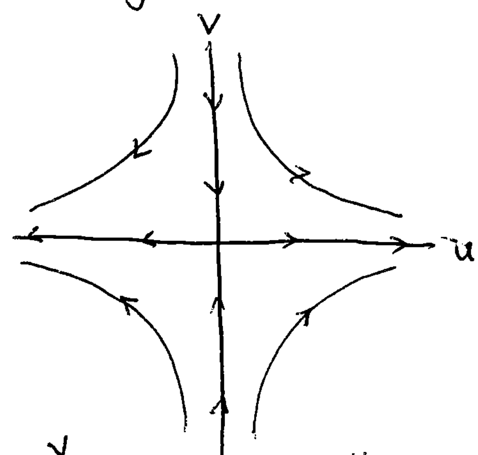
$$\frac{du}{dt} = 3u \qquad \frac{dv}{dt} = -v$$

where (u, v) are the coordinates in a new coordinate system with basis vectors

$$\vec{b}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \vec{b}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

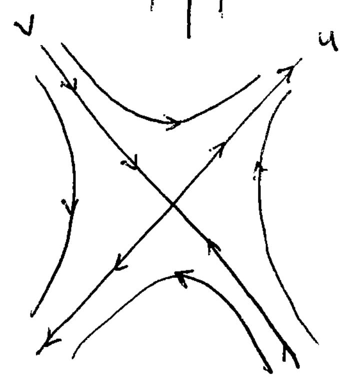
The equations are easily solved by integration. With the initial position $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$,

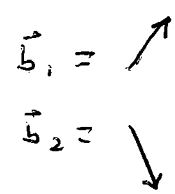
$$\boxed{\begin{aligned} u(t) &= u_0 e^{3t} \\ v(t) &= v_0 e^{-t} \end{aligned}}$$



The system grows exponentially along the u-axis and decays along the v-axis!

We can also re-draw this picture in the standard x-y coordinates, where the u and v axes look "tilted":





This is the same as the picture on page 1!

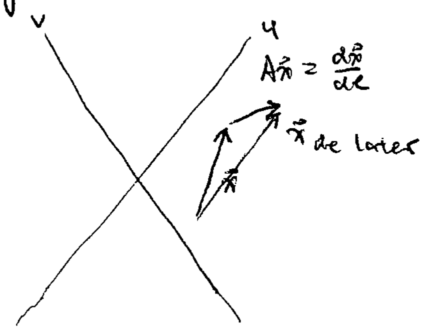
The moral of this story is: it's not that the problem is hard, but it seemed hard because the standard (x, y) coordinate system was the <u>wrong</u> set of coordinates. Here, the correct coordinates were (u, v), as defined by $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

How do we find the "right" coordinate system??

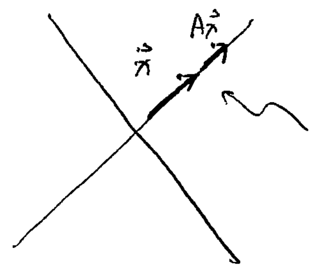
To see how to approach the problem, note from the phase plane that if we happen to lie across one of the "natural" axes, we stay on the axis for all future time because $\frac{d\vec{x}}{dt}$ and $\vec{x}$ lie in the same direction. First we rewrite the system of DEs in a matrix vector format:

$$\frac{dx}{dt} = x + y \qquad \frac{dy}{dt} = 4x + y \quad \Rightarrow \quad \underbrace{\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix}}_{\frac{d\vec{x}}{dt}} = \underbrace{\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{x}}$$

In this case, A is the rule that tells you how $\vec{x}$ changes after a time dt.



But if $\vec{x}$ lies along the u or v axis:



$\vec{x}$ is confined to the u axis for all future t since $\frac{d\vec{x}}{dt}$ is parallel to $\vec{x}$.

We can take advantage of this property to find what the special axes are. When we have $\vec{x}$ that lies on such axes, the following is true:

$$\boxed{A\vec{x} = \lambda\vec{x}}$$

Where λ is a number that tells you how much $A\vec{x}$ is stretched relative to $\vec{x}$.

λ - eigenvalue of A

$\vec{x}$ - eigenvector of A

For a given matrix A, we can solve for its eigenvalues and eigenvectors algebraically:

$$A\vec{x} = \lambda\vec{x}$$

$$A\vec{x} - \lambda \underbrace{I}_{\substack{\text{identity}\\ \text{matrix}}} \vec{x} = 0$$

$$(A - \lambda I)\vec{x} = 0$$

which will hold when the determinant of $(A - \lambda I)$ is 0:

$$\det(A - \lambda I) = 0$$

For $\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$, we have

$$\det\left(\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 0$$

$$\det \begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)(1-\lambda) - 4 = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda - 3)(\lambda + 1) = 0 \quad \Rightarrow \quad \begin{aligned} \lambda_1 &= 3 \\ \lambda_2 &= -1 \end{aligned}$$

We have two eigenvalues. We can find a separate eigenvector for each eigenvalue:

$$(A - \lambda I)\vec{x} = 0$$ Solve for $\vec{x}$ for each particular λ.

$\lambda_1 = 3$:

$$\begin{bmatrix} 1-3 & 1 \\ 4 & 1-3 \end{bmatrix} \vec{x}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\vec{x}_1$ & $\vec{x}_2$ are precisely the basis vectors for the new coordinate system!

$\lambda_2 = -1$

$$\begin{bmatrix} 1-(-1) & 1 \\ 4 & 1-(-1) \end{bmatrix} \vec{x}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\vec{x}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

To summarize, we've taken a coupled system of linear differential equations:

$$\frac{dx}{dt} = x + y \qquad \text{or} \qquad \frac{d\vec{x}}{dt} = \underbrace{\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{x}}_{A\vec{x}}$$
$$\frac{dy}{dt} = 4x + y$$

We found the eigenvalues and eigenvectors of A

$$\lambda_1 = 3 \;, \quad \vec{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \& \quad \lambda_2 = -1 \;, \quad \vec{x}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

We made a change of coordinates to the basis defined by the eigenvectors (a.k.a. eigenbasis):

$$\underbrace{P = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}}_{\substack{\text{change from new} \\ \text{eigenbasis to old}}} \qquad \underbrace{P^{-1} = \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}}_{\substack{\text{change from old} \\ \text{basis to eigenbasis}}} \qquad \vec{x}_{new} = P^{-1}\vec{x}$$

In this eigenbasis, the system is defined by a decoupled system:

$$\frac{du}{dt} = 3u \qquad \text{or} \qquad \frac{d\vec{x}_{new}}{dt} = \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}}_{D} \underbrace{\vec{x}_{new}}_{P^{-1}\vec{x}}$$
$$\frac{dv}{dt} = -v$$

Where D is the analog of A in the eigenbasis. Note that since D describes the decoupled system, it has zeroes everywhere except the diagonal, and the diagonals are the eigenvalues of A. D is said to be a <u>diagonal matrix</u>.

Finally, we can transform back to the original basis:

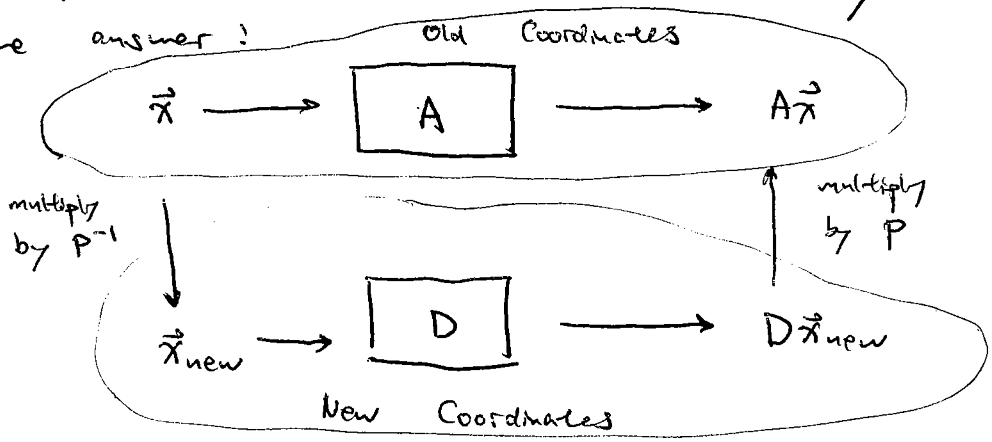
$$\frac{d\vec{x}}{dt} = \underbrace{\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}}_{P} \frac{d\vec{x}_{new}}{dt}$$

$$= \underbrace{\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}}_{D} \underbrace{\frac{1}{4}\begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}}_{P^{-1}} \vec{x}$$

$$\frac{d\vec{x}}{dt} = PDP^{-1}\vec{x}$$

If you do the matrix multiplication, you can confirm that $PDP^{-1} = A$. The process of "factoring" A into PDP^{-1} is called <u>diagonalization</u>.

Think of it as kind of a roundabout way to get to the answer!



This circular approach is necessary because the original formulation as the coupled system can't be solved directly.

> Note: D captures all the <u>qualitative</u> information about A. Because D is comprised of the eigenvalues of A, knowing the eigenvalues alone is sufficient for qualitative analysis of fixed points.