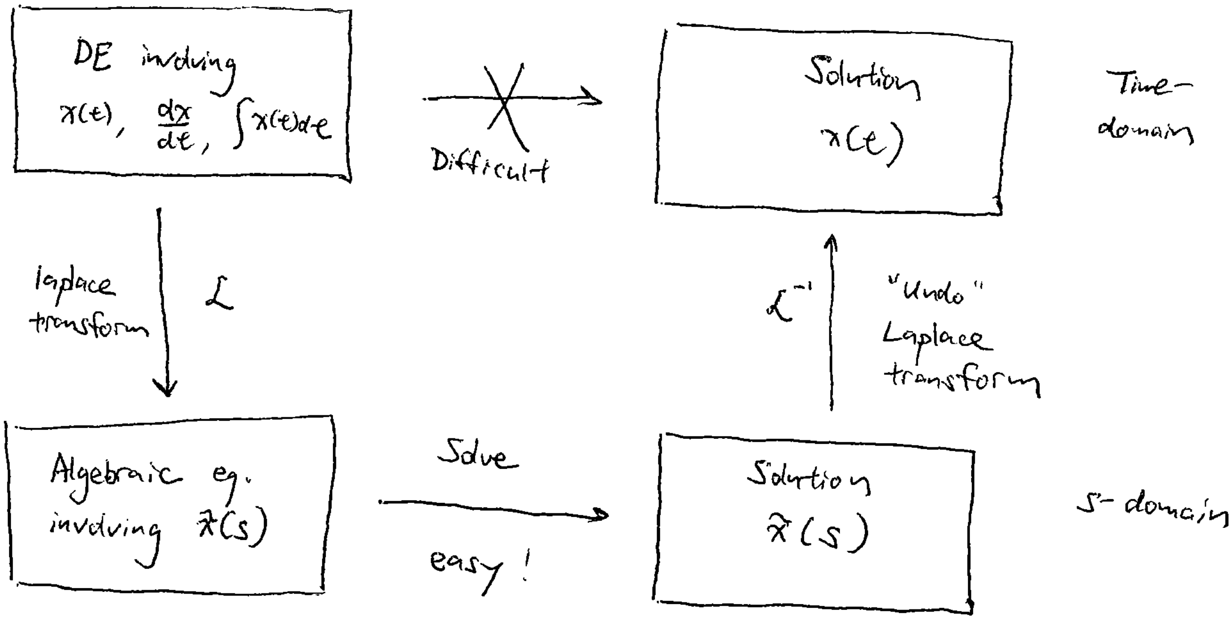
BP297
5/10/04

Section Notes

Motivation: Turn a hard problem (equation containing derivatives & integrals) into an equivalent but easy problem (algebraic equation.)



Definition:

$$\boxed{\mathcal{L}(f(t)) = \tilde{f}(s) = \int_0^\infty e^{-st} f(t)\,dt}$$

Transforming Constants:

$f(t) = 1$

$\tilde{f}(s) = 1/s$

$$\mathcal{L}(1) = \int_0^\infty e^{-st}\cdot 1\,dt = \int_0^\infty e^{-st}\,dt = \left[-\frac{e^{-st}}{s}\right]_0^\infty = \frac{1}{s}$$

Transforming Exponentials:

$f(t) = e^{at}$

$\tilde{f}(s) = \frac{1}{s-a}$

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{-st} e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \left[-\frac{e^{-(s-a)t}}{s-a}\right]_0^\infty = \frac{1}{s-a}$$

Transforming Derivatives:

$$\mathcal{L}(f'(t)) = \int_0^\infty e^{-st} f'(t)\,dt$$

Integrate by Parts: $\int u\,dv = uv - \int v\,du$

$u = e^{-st}$ $\quad dv = f'(t)\,dt$

$du = -se^{-st}$ $\quad v = f(t)$

$$= e^{-st} f(t) \Big|_0^\infty + s \int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + s\,\mathcal{L}\{f(t)\}$$

$$= s\tilde{f}(s) - f(0)$$

All we have to do is multiply by s!

Transforming Integrals:

$$\mathcal{L}\left\{\int_0^t f(t')\,dt'\right\} = \frac{\tilde{f}(s)}{s}$$

Integrals are just anti-derivatives

"Undoing" Laplace Transforms

In theory:
$$f(t) = \int_{-\infty i}^{\infty i} \frac{1}{2\pi i} \hat{f}(s) e^{st} ds$$

But as this involves complex integration, we will not apply it.

In practice: Build a table of $f(t)$ and their corresponding $\hat{f}(s)$, and lookup.

Example: Partial fractions

Find the $f(t)$ that, when Laplace transformed, gives
$$\hat{f}(s) = \frac{2s - 14}{s^2 - 2s - 3}$$

Problem takes the form $\frac{\text{polynomial}_1}{\text{polynomial}_2}$ $\Rightarrow$ apply method of partial fractions

1. Factor denominator:
$$\hat{f}(s) = \frac{2s - 14}{(s+1)(s-3)}$$

2. Express as sum of two fractions with unknown numerators:
$$\hat{f}(s) = \frac{A}{(s+1)} + \frac{B}{(s-3)}$$

3. Multiply by common denominator:
$$\hat{f}(s) = \frac{A(s-3) + B(s+1)}{(s+1)(s-3)}$$

4. Solve for A, B
$$A(s-3) + B(s+1) = 2s - 14$$
$$As + Bs = 2s$$
$$-3A + B = -14$$
$$\Rightarrow A = 4, \quad B = -2$$

$$\hat{f}(s) = \frac{4}{s+1} - \frac{2}{s-3}$$

Lookup each partial fraction:

$$f(t) = 4\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] - 2\mathcal{L}^{-1}\left[\frac{2}{s-3}\right]$$

$$= 4e^{-t} - 2e^{3t}$$

<u>Analyzing Long-term behavior</u> $(t \to \infty)$ of $f(t)$ & $\hat{f}(s)$

For $f(t) = 4e^{-t} - 2e^{3t}$,

we see that the e^{3t} term will dominate,

so $\lim_{t\to\infty} f(t) = -\infty$

We can predict the same behavior by looking at $\hat{f}(s)$

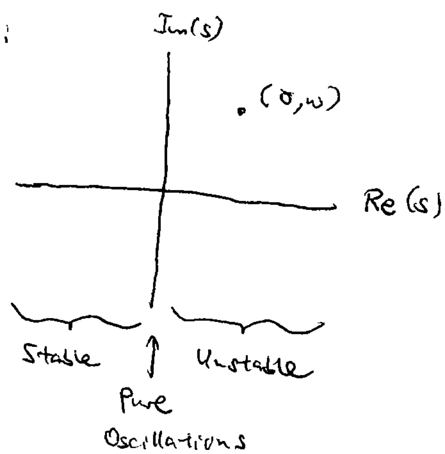
$$\hat{f}(s) = \frac{2s-14}{(s+1)(s-3)}$$

at $s=-1$ & $s=3$, denominator $= 0$ "blows-up"

$\Rightarrow$ poles of $\hat{f}(s)$

In general, $s = \sigma + i\omega$ can be complex. Plot s in complex plane:

When s has $\sigma > 0$, $f(t)$ goes to ∞ or $-\infty$ in the long run



Final Value Theorem: When all poles are on the left side of the complex plane:

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} s\,\hat{x}(s)$$